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α-Stable Limit Theorems for Sums of Dependent Random Vectors

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Several α-stable limit theorems for sums of dependent random vectors are proved via point processes theory; ρ-mixing, m-dependence, and the type of mixing treated within the extreme value theory are considered. © 1989 Academic Press, Inc.

1. INTRODUCTION

There exists a huge literature on the central limit problem for sums of dependent random variables while the weak convergence of such sums to the other laws has not been investigated so intensively.

In the present paper we examine convergence in distribution of sums of dependent random vectors to α-stable laws, $0<\alpha<2$. We prove Theorem 4.1, a finite-dimensional and nonstationary generalization of Davis' theorem [5, Theorem 1], next Theorem 4.2 which is a counterpart of Ibragimov's central limit theorem for ρ-mixing sequences [13], and also a limit theorem for partial sums of m-dependent stationary sequence (Theorem 5.3) which corresponds to the central limit theorem of Diananda [7].

As a main tool we use Theorem 3.1 on the weak convergence of sums to a generalized Poisson distribution, formulated in array setting without any assumption on stationarity. In its proof we apply the point processes method in a way similar to the approach of Durrett and Resnick [8, Section 4]. However, the criterion which guarantees the convergence in distribution of point processes is different: we apply the general theory of point processes due to Kallenberg [15] similarly as it has been done in the extreme value limit theory [17], i.e., considering modified Leadbetter's

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conditions D and D' [16], while Durrett and Resnick applied Freedman's theorem [9] based on Jager's theory of point processes [14].

Recently Resnick in [22] has given a systematic treatment of the application of point processes theory in various limit theorems for sequences of independent random variables (also for partial sums). It is possible to derive similar theorems in the case of dependent random variables making use of the convergence of point processes described in Remark 3.6; here, however, we restrict our considerations to the convergence of partial sums.

In Theorems 4.1 and 4.2 we assume the condition D'_0 which excludes clusters of big values in the rows of an array. This is a rather strong restriction since there exist some natural examples of sequences (e.g., moving averages), which do not have this property. However, such sequences (arrays) often can be replaced by some other ones, which lead to similar sums and satisfy the condition D'_0. We give an example of such a reduction in Sections 5 and 6, where we obtain limit theorems for sums of m-dependent stationary sequences.

2. Preliminaries

In what follows we need some conventions. Generally, we use the notation of the book of Araujo and Giné [2]. In particular, for $0 < \tau < +\infty$ $c_\tau - \text{Pois}(\nu)$ is a distribution on R^d, given by the characteristic function

$$\widehat{(c_\tau - \text{Pois}(\nu))}(y) = \exp\left[\int (e^{i(y,x)} - 1 - i(y,x)\,1(\|x\| \leqslant \tau))\,\nu(dx)\right], \tag{2.1}$$

where ν is a Lévy measure on R^d.

We also introduce

$$c_\infty - \text{Pois}(\nu) := (c_1 - \text{Pois}(\nu)) * \delta_{b_1}, \tag{2.2}$$

where

$$b_1 = -\int x1(\|x\| > 1)\,\nu(dx) \qquad \text{if} \quad \int \|x\|\; 1(\|x\| > 1)\,\nu(dx) < +\infty,$$

$$c_0 - \text{Pois}(\nu) := (c_1 - \text{Pois}(\nu)) * \delta_{b_2}, \tag{2.3}$$

where

$$b_2 = \int x1(\|x\| \leqslant 1)\,\nu(dx) \qquad \text{if} \quad \int \|x\|\; 1(\|x\| \leqslant 1)\,\nu(dx) < +\infty$$

(here "δ_b" denotes a probability measure concentrated in the point b, "$*$"—convolution, $1(B)$—the indicator of the set B, $\|\ \|$—norm in R^d).

The corresponding characteristic functions are of the form

$$\widehat{(c_\infty - \mathrm{Pois}(\nu))}(y) = \exp\left[\int (e^{i(y,x)} - 1 - i(y,x))\, \nu(dx)\right]$$

$$\widehat{(c_0 - \mathrm{Pois}(\nu))}(y) = \exp\left[\int (e^{i(y,x)} - 1)\, \nu(dx)\right].$$

A distribution μ on R^d is α-stable, $0<\alpha<2$, iff it has the representation

$$\mu = \delta_b * (c_{\tau(\alpha)} - \mathrm{Pois}(\nu(\alpha,\sigma))), \tag{2.4}$$

where $b \in R^d$ and $\nu(\alpha,\sigma)$ is the Lévy measure given by the formula

$$\nu(\alpha,\sigma)(A) = \int_{S^{d-1}} \int_0^\infty 1(A)(r,s) r^{-1-\alpha}\, dr\sigma(ds) \tag{2.5}$$

for all Borel subsets of the space $E^d = R^d \setminus \{0\}$. Here $S^{d-1} = \{x \in R^d; \|x\| = 1\}$, σ is a finite measure on S^{d-1} and $(r,s) \in (R^+\setminus\{0\}) \times S^{d-1}$ is an obvious parametrization of the space E^d. The function $\tau(\alpha)$ in (2.4) is defined by the formula

$$\tau(\alpha) = \begin{cases} 0 & \text{if } 0<\alpha<1 \\ 1 & \text{if } \alpha=1 \\ \infty & \text{if } 1<\alpha<2 \end{cases} \tag{2.6}$$

(see [2, p. 149]).

Let $\{X_k;\ k \in Z\}$ be a two-sided sequence of i.i.d. random vectors with values in R^d. The classical limit theory for independent summands asserts that one can find centering vectors b_n, and normalizing constants a_n, $n \in N$, such that the sequence

$$(S_n - b_n)/a_n = \left(\sum_{k=1}^n X_k - b_n\right)\Big/ a_n, \qquad n \in N,$$

converges in distribution to some nondegerated limit μ if and only if μ is an α-stable distribution $(0<\alpha\leqslant 2)$ and the marginal distribution $\mathscr{L}(X_0)$ belongs to the domain of attraction of μ:

$$\mathscr{L}(X_0) \in D(\mu).$$

If $0<\alpha<2$, then for $\mathscr{L}(X_0)$ be in $D(\mu)$ it is necessary and sufficient that $\mathscr{L}(X_0)$ varies regularly with index $(-\alpha)$ (see [10, 18]). In particular, if $\mathscr{L}(X_0) \in D(\mu)$ and μ is α-stable, $0<\alpha<2$, then $t^\alpha P(\|X_0\| > t)$ is a slowly varying function [24].

We finish this section with some remarks on Lévy processes refering for further discussion and literature to [22]. For every generalized Poisson distribution (in fact, for every infinitely divisible distribution) there exists a time-homogeneous process with independent increments $\{Y(s)=(y_1(s), y_2(s), ..., y_d(s))'; s \in [0, 1]\}$ such that $Y(0) \equiv 0$ and $\mathscr{L}(Y(1)) = c_\tau - \text{Pois}(\nu)$. Moreover, we can assume that the trajectories of the process $Y = \{Y(s); s \in [0, 1]\}$ are right-continuous and admit left limits. The Lévy measure ν is then the jump measure of Y, i.e., for every Borel subset A of E^d

$$\nu(A) = E\left(\sum_{s \leqslant 1} 1(\Delta Y(s) \in A) \right), \tag{2.7}$$

where $\Delta Y(s) = Y(s) - Y(s-)$. Let $C\colon R^d \to R^{d'}$ be a linear map of R^d onto $R^{d'}$ for some $0 \leqslant d' \leqslant d$. The process $\{C \circ Y(s), s \in [0, 1]\}$ still has independent increments and its Lévy measure ν_C is given by the formula

$$\nu_C(A) = \nu \circ C^{-1}(A) \tag{2.8}$$

for every Borel subset A of $E^{d'}$. In particular, for a Lévy measure ν on $(E^d)^p$ and for a finite and non-empty subset $I \subset \{1, 2, ..., p\}$ we obtain the Lévy measure

$$\nu_{\sum_{i \in I} x_i} = \nu \circ C_I^{-1} \tag{2.9}$$

where $C_I\colon R^{d \cdot p} \to R^d$, $C_I(x_1, ..., x_p) = \sum_{i \in I} x_i$.

3. Generalized Poisson Laws as Limits of Sums of Dependent Random Vectors

In [22] Resnick has shown that the theory of point processes can be very useful in the proofs of limit theorems for arrays of random vectors independent in rows. The idea is as follows: prove the convergence in distribution of certain point processes connected with the array of random vectors and then obtain such a convergence for compositions of those processes with some a.s. continuous functionals by the continuous mapping theorem [3, Theorem 5.1]. In the case of sums it is convenient to separate the essential part of each random vector and use, e.g., [3, Theorem 4.2]. Such a method for sums of dependent random variables was used for the first time by Durrett and Resnick in [8]. The result was formulated in terms of conditional quantities. In this section we prove a limit theorem for certain point processes connected with the array of random vectors via Kallenberg's theory [15] using some assumptions drawn from the extreme value theory (see, e.g., [17]).

Let $\{X_{nk},\ 1\leqslant k\leqslant k_n,\ n\in N\}$ be a double array of random vectors with values in R^d. Denote

$$S_n=\sum_{k=1}^{k_n} X_{nk},$$

$$S_n(]\eta,\delta])=\sum_{k=1}^{k_n} X_{nk}1(\eta<\|X_{nk}\|\leqslant\delta).$$

The array $\{X_{nk},\ 1\leqslant k\leqslant k_n,\ n\in N\}$ satisfies

Condition A_0 iff $\{X_{nk}\}$ is uniformly infinitesimal, i.e.

$$\max_{1\leqslant k\leqslant k_n} P(\|X_{nk}\|>\varepsilon)\to 0 \qquad \text{as} \quad n\to+\infty \text{ for every } \varepsilon>0. \tag{3.1}$$

Condition B_0 iff for every set S which is a finite sum of disjoint and separated from 0 sets of the form $\mathsf{X}_{i=1}^{d}\ ]a_i, b_i]$,

$$\begin{aligned}\sup|P(X_{nk}\in S^c,\ p<k\leqslant r)-P(X_{nk}\in S^c,\ p<k\leqslant q)\\ \times P(X_{nk}\in S^c,\ q<k\leqslant r)|\xrightarrow[\text{as } n\to\infty]{} 0,\end{aligned} \tag{3.2}$$

where supremum is taken over the set of all p, q, r such that $0\leqslant p\leqslant q\leqslant r\leqslant k_n$,

Condition D_0' iff

$$\lim_{r\to\infty}\limsup_{n\to\infty}\inf_{\Delta(r)}\sum_{q=1}^{r}\sum_{m_{q-1}<i<j\leqslant m_q} P(\|X_{ni}\|>\varepsilon,\ \|X_{nj}\|>\varepsilon)=0, \tag{3.3}$$

where $\Delta(r)$ is an arbitrary division of the set $\{1, 2, ..., k_n\}$ into r segments (possibly empty) $0=m_0\leqslant m_1\leqslant\cdots\leqslant m_r=k_n$ (we use the notation $\sum_\phi=0$, $\bigcap_\phi=\Omega$).

The property B_0 is a stronger version of Leadbetter's condition D, introduced by Davis during examination of limit laws for order statistics [4] and D_0' is one of the generalizations of Leadbetter's condition D′ (see also [12, 19]).

THEOREM 3.1. *Let the array* $\{X_{nk};\ 1\leqslant k\leqslant k_n,\ n\in N\}$ *satisfy* A_0, B_0, *and* D_0'. *Assume that there exists an atomless measure* ν *on* E^d, *finite outside each neighbourhood of* 0 *and such that*

$$\sum_{k=1}^{k_n} P(X_{nk}\in A)\to\nu(A), \qquad n\to\infty \tag{3.4}$$

for every element A *of a determining class in* E^d *(in the sense of* [3, p. 15]*).*

(i) *If*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \sum_{k=1}^{k_n} E(\|X_{nk}\| \, 1(\|X_{nk}\| \leqslant \delta)) = 0 \tag{3.5}$$

then

$$\int \|x\| \, 1(\|x\| \leqslant 1) \, \nu(dx) < +\infty \quad \textit{and} \quad S_n \xrightarrow[\mathscr{D}]{} \mathrm{Pois}(\nu).$$

(ii) *If ν is a Lévy measure (i.e., $\int \min(1, x^2) \, \nu(dx) < +\infty$) and*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P(\|S_n(]0, \delta]) - ES_n(]0, \delta])\| > \varepsilon)$$
$$= 0 \qquad \textit{for every} \quad \varepsilon > 0 \tag{3.6}$$

then

$$S_n - ES_n(]0, \tau]) \xrightarrow[\mathscr{D}]{} c_\tau - \mathrm{Pois}(\nu)$$

for every $\tau > 0$ such that $\nu(\|x\| = \tau) = 0$.

(iii) *If ν is a Lévy measure,* (3.6) *is satisfied and additionally*

$$\lim_{\lambda \to \infty} \limsup_{n \to \infty} \sum_{k=1}^{k_n} E(\|X_{nk}\| \, 1(\|X_{nk}\| > \lambda)) = 0 \tag{3.7}$$

then

$$S_n - ES_n \xrightarrow[\mathscr{D}]{} c_\infty - \mathrm{Pois}(\nu).$$

(*The symbol* "$\to_{\mathscr{D}}$" *denotes convergence in distribution.*)

Remark. In the sequel we will use Theorem 3.1 for the Lévy measures $\nu(\alpha)$ of α-stable distributions, $0 < \alpha < 2$. Obviously $\nu(\alpha)$, $0 < \alpha < 2$, are atomless.

In the proof of Theorem 3.1 we need the following lemma.

LEMMA 3.2. *Let $\{A_{nk};\ 1 \leqslant k \leqslant k_n,\ n \in N\}$ be an array of events defined on a probability space $(\Omega, \mathscr{F}, P)$. Assume that the array satisfies the following conditions*:

Condition $\bar{\mathrm{A}}_0$.

$$\min_{1 \leqslant k \leqslant k_n} P(A_{nk}) \to 1, \qquad n \to \infty \tag{3.8}$$

Condition $\bar{\mathrm{B}}_0$.

$$\sup\left|P\left(\bigcap_{j=p+1}^{r} A_{nj}\right)-P\left(\bigcap_{j=p+1}^{q} A_{nj}\right)P\left(\bigcap_{j=q+1}^{r} A_{nj}\right)\right|\to 0, \qquad n\to\infty, \tag{3.9}$$

where supremum is taken over the set of all p, q, $r\in N$ *such that* $0\leqslant p\leqslant q\leqslant r\leqslant k_n$.

Condition $\bar{\mathrm{D}}_0'$.

$$\lim_{r\to\infty}\limsup_{n\to\infty}\inf_{\Delta(r)}\sum_{q=1}^{r}\sum_{m_{q-1}<i<j\leqslant m_q} P(A_{ni}^c\cap A_{nj}^c)=0, \tag{3.10}$$

where $\Delta(r)$ *is the same as in* D_0'. *Then*

$$P\left(\bigcap_{k=1}^{k_n} A_{nk}\right)-\exp\left(-\sum_{k=1}^{k_n} P(A_{nk}^c)\right)\to 0, \qquad n\to\infty. \tag{3.11}$$

Proof of Lemma 3.2. It is sufficient to show that under $\bar{\mathrm{A}}_0$, $\bar{\mathrm{B}}_0$, $\bar{\mathrm{D}}_0'$ the convergence

$$\sum_{k=1}^{k_n'} P(A_{nk}^c)\to C, \qquad n\to\infty \text{ (where } C\in[0,+\infty])$$

implies the convergence

$$P\left(\bigcap_{k=1}^{k_n} A_{nk}\right)\to e^{-C} \qquad (\text{where } e^{-\infty}=0) \tag{3.12}$$

for every subsequence $\{k_n',\ n\in N\}$ of the sequence $\{k_n\}$. In the sequel we write for brevity k_n instead of k_n'.

If $C=0$, then (3.12) follows trivially. Assume that $0<C<+\infty$. Fix $r\in N$ and define

$$j_{n,0}^r=0;$$

$$j_{n,p}^r=\begin{cases}\inf\{k;\ \sum_{s=1}^{k} P(A_{ns}^c)\geqslant pC/r\} & \text{if the set is not empty}\\ k_n & \text{otherwise;}\end{cases}$$

for positive integers p such that $0<p<r$,

$$j_{n,r}^r=k_n.$$

By the property $\bar{\mathrm{A}}_0$ for every $p\in\{1, 2, ..., r\}$,

$$\sum_{k=j_{n,p-1}^r+1}^{j_{n,p}^r} P(A_{nk}^c)\to C/r, \qquad n\to\infty \tag{3.13}$$

and by the property $\bar{\mathrm{B}}_0$ for fixed r,

$$P\left(\bigcap_{1\leqslant k\leqslant k_n} A_{nk}\right)-\prod_{p=1}^{r} P\left(\bigcap_{k=j^r_{n,p-1}+1}^{j^r_{n,p}} A_{nk}\right)\to 0, \quad n\to\infty. \tag{3.14}$$

Let $\{N_r, r\in N\}$ be such that $N_r>N_{r-1}$ and for $n\geqslant N_r$,

$$\max_{1\leqslant p\leqslant r}\left|\sum_{k=j^r_{n,p-1}+1}^{j^r_{n,p}} P(A^c_{nk})-C/r\right|\leqslant 1/r \tag{3.15}$$

$$\left|P\left(\bigcap_{1\leqslant k\leqslant k_n} A_{nk}\right)-\prod_{p=1}^{r} P\left(\bigcap_{k=j^r_{n,p-1}+1}^{j^r_{n,p}} A_{nk}\right)\right|\leqslant 1/r. \tag{3.16}$$

For natural n, define

$$r_n:=r \qquad \text{iff} \quad N_r\leqslant n<N_{r+1}.$$

Clearly, $r_n\to\infty$ as $n\to\infty$. Let

$$m_{n,p}:=j^{r_n}_{n,p}, \qquad p=0, 1, ..., r_n.$$

Divide the intersection $\bigcap_{1\leqslant k\leqslant k_n} A_{ni}$ into r_n blocks

$$B_{np}=\bigcap_{k=m_{n,p-1}+1}^{m_{n,p}} A_{nk}, \qquad p=1, 2, ..., r_n \tag{3.17}$$

(here $\bigcap_\phi=\Omega$). We have by (3.16) and (3.15)

$$P\left(\bigcap_{p=1}^{r_n} B_{np}\right)-\prod_{p=1}^{r_n} P(B_{np})\to 0, \qquad n\to\infty \tag{3.18}$$

$$\max_{1\leqslant p\leqslant r_n} P(B^c_{np})\to 0, \qquad n\to\infty. \tag{3.19}$$

Since $|\exp(-x)-1+x|\leqslant(1/2)x^2$ for $x\geqslant 0$, so

$$\left|\prod_{p=1}^{r_n} P(B_{np})-\exp\left(\sum_{p=1}^{r_n} P(B_{np})-1\right)\right|$$

$$\leqslant\tfrac{1}{2}\max_{1\leqslant p\leqslant r_n} P(B^c_{np})\left(\sum_{p=1}^{r_n} P(B^c_{np})\right)\to 0, \quad n\to\infty. \tag{3.20}$$

Hence

$$\liminf_{n\to\infty} P\left(\bigcap_{k=1}^{k_n} A_{nk}\right)\geqslant\exp\left(-\liminf_{n\to\infty}\sum_{k=1}^{k_n} P(A^c_{nk})\right)=\exp(-C). \tag{3.21}$$

Now we fix $r \in N$, choose the division Δ'_n of the set $\{1, 2, ..., k_n\}$, $\Delta'_n = \{0 = m'_{n0} \leqslant m'_{n1} \leqslant \cdots \leqslant m'_{nr} = k_n\}$ and define the blocks

$$C^r_{np} = \bigcap_{k = m'_{n,p-1}+1}^{m'_{n,p}} A_{nk}.$$

By the Bonferroni inequality,

$$-\sum_{p=1}^{r} P((C^r_{np})^c) \leqslant -\sum_{k=1}^{k_n} P(A^c_{nk}) + \sum_{p=1}^{r} \sum_{m'_{n,p-1} < i < j \leqslant m'_{n,p}} P(A^c_{ni} \cap A^c_{nj}). \quad (3.22)$$

Due to $\bar{\mathrm{D}}'_0$ the right-hand side of (3.22) can be made arbitrarily close to $(-C)$ if only r and Δ'_n are properly chosen. This together with the inequality $\exp(-x) \geqslant 1 - x$ (for $x \geqslant 0$) gives (due to $\bar{\mathrm{B}}_0$)

$$\begin{aligned} \limsup_{n \to \infty} P\left(\bigcap_{k=1}^{k_n} A_{nk}\right) &= \limsup_{n \to \infty} \prod_{p=1}^{r} P(C^r_{np}) \\ &\leqslant \limsup_{n \to \infty} \exp\left[-\sum_{p=1}^{r} P((C^r_{np})^c)\right] \\ &\leqslant \exp(-C). \end{aligned} \quad (3.23)$$

The inequalities (3.21) and (3.23) prove (3.11) for $0 < C < +\infty$. If $C = +\infty$, then the estimation (3.22) gives under $\bar{\mathrm{B}}_0$ and $\bar{\mathrm{D}}'_0$ the convergence $P(\bigcap_{k=1}^{k_n} A_{nk}) \to 0$, $n \to \infty$. ∎

For an array $\{X_{nk},\ 1 \leqslant k \leqslant k_n,\ n \in N\}$ of random vectors we define a sequence $\{N_n, n \in N\}$ of point processes on E^d as

$$N_n(A) = \sum_{k=1}^{k_n} 1(X_{nk} \in A), \qquad A \subset E^d.$$

Let Π_ν be a Poisson process on E^d with intensity ν (for definition and properties see [15, 22]).

LEMMA 3.3. *Conditions* A_0, B_0, D'_0 *and* (3.4) *imply the convergence in distribution of processes* N_n *to* Π_ν:

$$N_n \xrightarrow[\mathscr{D}]{} \Pi_\nu. \quad (3.24)$$

Remark 3.4. We treat the processes N_n, Π_ν as the measurable mappings of $(\Omega, \mathscr{F}, P)$ into the space $\mathscr{N}$ described as follows. Let $\mathscr{M} = \mathscr{M}(E^d)$ be the space of locally finite measures on the Borel sets in E^d (i.e., $\mu \in \mathscr{M}$ iff $\mu(K) < +\infty$ for all compact subsets $K \subset E^d$). $\mathscr{M}$ is a Polish space when considered with the vague convergence ($\mu_n \to \mu$ vaguely iff $\int f d\mu_n \to \int f d\mu$ for every continuous function f: $E^d \to R^1$ with compact support).

$\mathcal{N}$ is a subspace of $\mathcal{M}$ consisting of measures taking values in the set $\{0, 1, 2, ..., +\infty\}$. The convergence $N_n \to_{\mathcal{D}} \Pi_\nu$ denotes the convergence $P \circ N_n^{-1}(A) \to P \circ \Pi_\nu^{-1}(A)$ for all A belonging to the σ-algebra generated by the vague topology in $\mathcal{N}$ such that $\Pi_\nu(\partial A) = 0$ P-a.s. For the equivalent definitions we refer to [15, Theorem 4.2, Lemma 4.4]. In the proof of (3.24) we will use the following criterion which is an adaption (to the space E^d) of a general rule [15, Theorem 4.7].

LEMMA 3.5. *Let $\{\xi_n, n \in N\}$ be a sequence of point processes and ξ a simple point process (i.e., a point process with atoms of unit mass only) all defined on the space $(\Omega, \mathcal{F}, P)$ with values in E^d. Suppose that*

$$\limsup_{n \to \infty} E\xi_n(B) \leqslant E\xi(B) \tag{3.25}$$

for all bounded sets of the form $B = \mathsf{X}_{i=1}^d\,]a_i, b_i]$ such that $0 \notin \bar{B}$, $E\xi(B) < +\infty$, $\xi(\partial B) = 0$ a.s., and

$$\lim_{n \to \infty} P(\xi_n(S) = 0) = P(\xi(S) = 0) \tag{3.26}$$

for all the sets $S \subset E^d$ which are finite sums of disjoint blocks B used in (3.25). Then

$$\xi_n \xrightarrow[\mathcal{D}]{} \xi.$$

Proof of Lemma 3.3. The property (3.25) for $\{N_n, n \in N\}$ and Π_ν follows immediately from (3.4). Moreover, since

$$\{N_n(S) = 0\} = \bigcap_{k=1}^{k_n} \{X_{nk} \in S^c\},$$

so due to Lemma 3.2,

$$\begin{aligned}\lim_{n \to \infty} P(N_n(S) = 0) &= \lim_{n \to \infty} \exp(-EN_n(S)) \\ &= \exp(-\nu(S)) = P(\Pi_\nu(S) = 0).\end{aligned}$$

Lemma 3.5 gives (3.24). ∎

Proof of Theorem 3.1. Convergence (3.24) implies

$$\begin{aligned}S_n]\delta, \lambda] &= \int_{E^d} x 1(\delta < \|x\| \leqslant \lambda)\, N_n(dx) \\ &\xrightarrow[\mathcal{D}]{} \int_{E^d} x 1(\delta < \|x\| \leqslant \lambda)\, \Pi_\nu(dx) \end{aligned} \tag{3.27}$$

for every δ, $\lambda>0$, $\nu(\|x\|=\delta)=\nu(\|x\|=\lambda)=0$ while $\int_{E^d} x1(\delta<\|x\|\leqslant\lambda)$ $\Pi_\nu(dx)$ has a generalized Poisson distribution $\mathrm{Pois}(\nu|_{(\varepsilon<\|x\|\leqslant\lambda)})$. Due to (3.4) we have for $0<\varepsilon<\tau$ such that $\nu(\|x\|=\varepsilon)=\nu(\|x\|=\tau)=0$,

$$ES_n]\varepsilon,\tau]\to\int x1(\varepsilon<\|x\|\leqslant\tau)\,\nu(dx). \tag{3.28}$$

We prove each of the three parts of the thesis separately, using [3, Theorem 4.2] every time:

(i) Consider the array $\{S_n]\varepsilon_j,\lambda_j];\ j\leqslant k_n,\ n\in N\}$, where $\varepsilon_j\searrow 0$, $\lambda_j\nearrow+\infty$ and $\nu(\|x\|=\varepsilon_j)=\nu(\|x\|=\lambda_j)=0$ for $j\in N$. For fixed j we have by (3.27),

$$S_n]\varepsilon_j,\lambda_j]\xrightarrow[\mathscr{D}]{}\mathrm{Pois}(\nu|_{(\varepsilon_j<\|x\|\leqslant\lambda_j)}). \tag{3.29}$$

Moreover, by (3.4) and (3.5),

$$\lim_{j\to\infty}\limsup_{n\to\infty}P(\|S_n]\varepsilon_j,\lambda_j]-S_n\|\geqslant\eta)=0 \tag{3.30}$$

$$\int_0^1\|x\|\,\nu(dx)<+\infty. \tag{3.31}$$

Now the estimation

$$|\exp[i(t,x)]-1|\leqslant\|t\|\,\|x\|\,1(\|x\|\leqslant 1)+2\,1(\|x\|>1)$$

and the properties of the measure ν allow us to obtain, by the Lebesgue dominated theorem, the convergence

$$\int\exp[i(t,x)]-1)\,\nu|_{(\varepsilon_j<\|x\|\leqslant\lambda_j)}(dx)\to\int(\exp[i(t,x)]-1)\,\nu(dx). \tag{3.32}$$

(3.29), (3.30), (3.32), and [3, Theorem 4.2] give the convergence in thesis (i).

(ii) Consider the array

$$\{K_{nj};j\leqslant k_n,n\in N\}=\{S_n]\varepsilon_j,\lambda_j]-ES_n]\varepsilon_j,\tau];j\leqslant k_n,n\in N\},$$

where ε_j, λ_j are as in (i) and $\nu(\|x\|=\tau)=0$, $\varepsilon_j<\tau<\lambda_i$, $i,j\in N$. By (3.27) and (3.28) we have

$$K_{nj}\xrightarrow[\mathscr{D}]{}\delta_{b(\varepsilon_j,\tau)}*\mathrm{Pois}(\nu|_{(\varepsilon_j<\|x\|\leqslant\lambda_j)}), \tag{3.33}$$

where $b(\varepsilon_j, \tau) = -\int x 1(\varepsilon_j < \|x\| \leqslant \tau)\, \nu(dx)$. Moreover, by (3.6) and (3.4),

$$\lim_{j\to\infty} \limsup_{n\to\infty} P(\|K_{nj} - S_n\| > \eta) = 0 \quad \text{for every } \eta > 0. \tag{3.34}$$

It remains to show

$$c_\tau - \mathrm{Pois}(\nu|_{(\varepsilon_j < \|x\| \leqslant \lambda_j)}) \xrightarrow[w]{} c_\tau - \mathrm{Pois}(\nu) \tag{3.35}$$

("$\to_w$" denotes weak convergence of probability measures). But this can be obtained as in (i) from the Lebesgue dominated theorem for the sequence

$$\{(\exp[i(t, x)] - 1 - i(t, x)\, 1(\|x\| \leqslant \tau))\, 1(\varepsilon_j < \|x\| \leqslant \lambda_j);\ j \in N\},$$

since ν is a Lévy measure and

$$\begin{aligned}&|\exp[i(t, x)] - 1 - i(t, x)\, 1(\|x\| \leqslant \tau)|\\ &\qquad \leqslant \|t\|^2\, \|x\|^2\, 1(\|x\| \leqslant \tau) + 2\, 1(\|x\| > \tau).\end{aligned}$$

(iii) Consider the array

$$\{L_{nj}; j \leqslant k_n, n \in N\} = \{S_n]\varepsilon_j, \lambda_j] - ES_n]\varepsilon_j, \lambda_j]; j \leqslant k_n, n \in N\},$$

where ε_j, λ_j are as in (i). As in (i),

$$L_{nj} \xrightarrow[\mathscr{D}]{} \delta_{b(\varepsilon_j, \lambda_j)} * \mathrm{Pois}(\nu|_{(\varepsilon_j < \|x\| \leqslant \lambda_j)}), \quad n \to \infty, \tag{3.36}$$

where $b(\varepsilon_j, \lambda_j) = -\int x 1(\varepsilon_j < \|x\| \leqslant \lambda_j) \nu(dv)$. Moreover, ES_n, $n \in N$, exist due to (3.7) and by (3.6), (3.7), we have

$$\lim_{j\to\infty} \limsup_{n\to\infty} P(\|L_{nj} - (S_n - ES_n)\| > \eta) = 0 \quad \text{for every } \eta > 0. \tag{3.37}$$

In order to obtain the third assumption of [3, Theorem 4.2] we may apply the Lebesgue dominated theorem for the sequence

$$\{(\exp[i(t, x)] - 1 - i(t, x))\, 1(\varepsilon_j < \|x\| \leqslant \lambda_j), j \in N\},$$

since ν is a Lévy measure and the following inequalities hold:

$$\int \|x\|\, 1(\|x\| > 1)\, \nu(dx) < +\infty \qquad \text{by} \quad (3.7) \text{ and } (3.4),$$

$$\begin{aligned}&|\exp[i(t, x)] - 1 - i(t, x)|\\ &\qquad \leqslant \|t\|^2\, \|x\|^2\, 1(\|x\| \leqslant 1) + (2 + \|t\|\, \|x\|)\, 1(\|x\| > 1). \quad \blacksquare\end{aligned}$$

Remark 3.6. Let $\{X_{nk};\, k \in Z,\, n \in N\}$ be an array of stationary in rows random vectors defined on $(\Omega, \mathscr{F}, P)$ with values in $V \subset R^d$. Consider the sequences of point processes

$$I_n = \sum_{k=1}^{n} \delta_{(k/n,\, X_{nk})}, \qquad J_n = \sum_{k=1}^{\infty} \delta_{(k/n,\, X_{nk})}$$

which belong to $\mathscr{N}(]0, 1] \times V)$ and $\mathscr{N}(]0, +\infty[\times V)$, respectively. If there exists a measure ν on the Borel sets of V such that

$$nP(X_{n1} \in A) \to \nu(A), \qquad n \to \infty,$$

for A belonging to the determining class in V, then under some modifications of the conditions D_0 (as in [1]) and D_0' ($\Delta(r)$ is a division of the set $\{1, ..., [nT]\}$, $T \in N$) one can obtain the convergences

$$I_n \xrightarrow[\mathscr{D}]{} I, \qquad J_n \xrightarrow[\mathscr{D}]{} J, \quad n \to \infty,$$

where I and J are Poisson processes on $]0, 1] \times V$ and $]0, +\infty[\times V$ with the intensities $l|_{]0,1]} \times \nu$ and $l|_{]0,+\infty]} \times \nu$, respectively. Thus many theorems in [22] have their analogs in the dependent case. ∎

4. α-Stable Limit Theorems

Consider a sequence $\{X_k;\, k \in N\}$ of random vectors with equal marginal distributions, i.e.,

$$\mathscr{L}(X_k) = \mathscr{L}(X_1), \qquad k \in N, \tag{4.1}$$

defined on the probability $(\Omega, \mathscr{F}, P)$ with values in R^d.

Assume that the distribution $\mathscr{L}(X_1)$ belongs to the domain of attraction of the α-stable distribution μ_α, $0 < \alpha < 2$; i.e., there exist norming constants $\{a_n,\, n \in N\}$ and centering vectors $\{b_n \in R^d;\, n \in N\}$ such that

$$(\hat{S}_n - b_n)/a_n \xrightarrow[\mathscr{D}]{} \mu_\alpha, \qquad n \to \infty, \tag{4.2}$$

where $\hat{S}_n = \hat{X}_1 + \cdots + \hat{X}_n$ is a partial sum of the sequence $\{\hat{X}_k;\, k \in N\}$ "associated" to $\{X_k;\, k \in N\}$ (i.e., $\{\hat{X}_k;\, k \in N\}$ is an i.i.d. sequence of random vectors such that $\mathscr{L}(\hat{X}_1) = \mathscr{L}(X_1)$. If (4.2) holds then necessarily

$$h(t) = t^\alpha P(\|X_1\| > t) \text{ is a slowly varying function.} \tag{4.3}$$

Thus in the case $0<\alpha<1$, one can take $b_k=0$ for $k\in N$ in (4.2) and $\mu_\alpha=$ Pois $\nu(\alpha,\sigma)$.

THEOREM 4.1. *If* $0<\alpha<1$ *and* $\{a_n;\ n\in N\}$ *satisfies* (4.2) (*with, e.g.,* $b_n=0$, $n\in N$) *and the two following conditions are fulfilled*:

$$\sup_{0\leqslant p<q<r\leqslant n}\left|P\Big(\bigcap_{p<k\leqslant r}(X_k/a_n\in S^c)\Big)-P\Big(\bigcap_{p<k\leqslant q}(X_k/a_n\in S^c)\Big)\right.$$
$$\left.\times P\Big(\bigcap_{q<k\leqslant r}(X_k/a_n\in S^c)\Big)\right|\xrightarrow[n\to\infty]{}0 \tag{4.4}$$

for every $S\subset E^d$ *which is a finite sum of disjoint and separated from* 0 *sets of the form* $\mathsf{X}_{i=1}^d\,]a_i,b_i]$,

$$\lim_{r\to\infty}\limsup_{n\to\infty}\inf_{\Delta(r)}\sum_{q=1}^{r}\sum_{m_{q-1}<i<j\leqslant m_q}P(\|X_i\|>a_n\varepsilon,\ \|X_j\|>a_n\varepsilon)=0 \tag{4.5}$$

for every $\varepsilon>0$, *where* $\Delta(r)$ *is as in* (3.3), *then*

$$S_n/a_n\xrightarrow[\mathscr{D}]{}\mu_\alpha=\text{Pois }\nu(\alpha,\sigma). \tag{4.6}$$

Proof. We apply Theorem 3.1(i) for the array $\{X_{nk}=X_k/a_n;\ n\in N,\ k=1,...,n\}$. The assumptions A_0, (3.4), and (3.5) are implied by (4.2). The condition B_0 is true because of (4.4) and D_0' follows from (4.5). ∎

Theorem 4.1 is a generalization of [5, Theorem 1] in three aspects: we consider d-dimensional random vectors and not random variables, our sequence has identical one-dimensional marginal distributions and need not be stationary, and (4.5) is slightly weaker than the Davis' condition D′.

The following Theorem 4.2 can be considered as the α-stable counterpart of the Ibragimov's central limit theorem for ρ-mixing sequences [13]. For simplicity we assume $d=1$. A sequence of random variables $\{X_k;\ k\in N\}$ is ρ-mixing iff

$$\rho(n)=\sup_{k\in N}\sup_{m\in N}\{|\text{cor}(f,g)|;\ f\in L^2(\sigma(X_{k+1},...,X_{k+m})),$$
$$g\in L^2(\sigma(X_{k+m+n+1},...))\}\xrightarrow[n\to\infty]{}0. \tag{4.7}$$

THEOREM 4.2. *Consider a sequence of random variables satisfying* (4.1), (4.2), *and let* $\alpha\in[1,2[$. *If* (4.5) *is fulfilled*, $\{X_n,\ n\in N\}$ *is* ρ-*mixing with*

$$\sum_{i=1}^{\infty}\rho(2^i)<+\infty \tag{4.8}$$

and $\{a_n, n \in N\}$, $\{b_n, n \in N\}$ *are as in* (4.2), *then*

$$(S_n - b_n)/a_n \xrightarrow[\mathscr{D}]{} c_{\tau(\alpha)} - \mathrm{Pois}(\nu(\alpha, \sigma)) = \mu_\alpha,$$

where $\tau(\alpha)$ *is defined in* (2.6), $\nu(\alpha, \sigma)$ *in* (2.5).

Proof. We apply Theorem 3.1(ii), (iii) for the array

$$\{X'_{nk} = X_k/a_n;\ k = 1, ..., n,\ n \in N\}.$$

Conditions A_0, (3.4), (3.7) follow from (4.2), D'_0 from (4.5), and B_0 from A_0 and ρ-mixing. It remains to check (3.6). Denote

$$S'_n]a, b] = \sum_{k=1}^{n} X'_{nk}\, 1(a < |X'_{nk}| \leqslant b).$$

Due to (4.8) and [21, Lemma 3.4], there exists a constant K independent of $\delta > 0$ such that

$$\mathrm{Var}(S'_n]0, \delta]) \leqslant K \cdot n \cdot \mathrm{Var}(X'_{n1} 1(\|X'_{n1}\| \leqslant \delta)). \tag{4.9}$$

But since $\mathscr{L}(X_1)$ belongs to the domain of attraction of μ_α, so due to (4.3),

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathrm{Var}\, S'_n]0, \delta] = 0 \tag{4.10}$$

which gives (3.6). ∎

Remark 4.3. Theorem 4.2 is true if we assume stationarity of two-dimensional distributions and instead of (4.5) put the assumption

$$\bar{\Psi}^* = \limsup_{u \to +\infty} \sup_{k \geqslant 2} P(|X_1| > u, |X_k| > u)/P(|X_1| > u)^2 < +\infty. \tag{4.11}$$

Samur obtained as a corollary from a more general theorem the convergence $(S_n - b_n)/a_n \to_{\mathscr{D}} c_1 - \mathrm{Pois}\, \nu(\alpha, \sigma)$ for a stationary sequence $\{X_n, n \in N\}$ which is φ-mixing (i.e.,

$$\varphi(n) = \sup_{m \in N} \{|P(A/B) - P(A)|, B \in \sigma(X_1, ..., X_m),$$
$$A \in \sigma(X_{m+n+1}, ...)\} \xrightarrow[n \to \infty]{} 0)$$

and additionally fulfills

$$\varphi(1) < 1 \tag{4.12}$$

$$\sum_{n=1}^{\infty} \varphi^{1/2}(n) < +\infty \tag{4.13}$$

$$\Psi^* = \sup\{P(A \cap B)/P(A)\,P(B) \colon A \in \sigma(X_1, ..., X_m),$$
$$B \in \sigma(X_{m+1}, ...),\ P(A)\,P(B) > 0,\ m \in N\} < +\infty \tag{4.14}$$

[23, Corollary 5.10]. Hence we get an improvement over Samur's result, since we assume stationarity of two-dimensional distributions only, take ρ-mixing with the weak (4.8) instead of φ-mixing, (4.12) and (4.13); and, finally, (4.11) is sufficient instead of (4.14). ∎

5. Stable Distributions as Weak Limits of Partial Sums of a Strictly Stationary m-Dependent Sequence

The central limit theorem for stationary m-dependent sequences obtained in [11, 7, 20] has quite a satisfactory form.

This section is devoted to some investigations concerning the convergence in distribution of partial sums of m-dependent random vectors to α-stable limits for $0 < \alpha < 2$.

A two-sided sequence $\{X_k;\ k \in Z\}$ of random vectors is said to be m-dependent if, for every $n \in N$, the σ-algebras $\sigma(..., X_{n-1}, X_n)$ and $\sigma(X_{n+m+1}, X_{n+m+2}, ...)$ are independent.

The first lemma explains the properties A_0, B_0, and D_0' for an array of random vectors m-dependent in rows and this way shows what advantages can be gained by adapting Theorem 3.1 directly for such an array.

Lemma 5.1. *Assume that the array $\{X_{nk},\ 1 \leqslant k \leqslant k_n,\ n \in N\}$ of random vectors defined on some $(\Omega, \mathscr{F}, P)$ with values in R^d is m-dependent in rows and satisfies* A_0. *Then*

(i) *Condition* B_0 *is fulfilled.*

(ii) *If*

$$\sup_n \sum_{k=1}^{k_n} P(\|X_{nk}\| > \varepsilon) < c(\varepsilon) < +\infty, \quad \textit{for every } \varepsilon > 0, \tag{5.1}$$

then D_0' *is equivalent to the property* $D_0'(m)$

$$\lim_{r\to\infty} \limsup_{n\to\infty} \inf_{\Delta(r)} \sum_{q=1}^{r} \sum_{\substack{m_{q-1} < i < j \leqslant m_q \\ j-i \leqslant m}} P(\|X_{ni}\| > \varepsilon,\ \|X_{nj}\| > \varepsilon) = 0. \tag{5.2}$$

Proof. We show only that (5.1) and (5.2) imply D_0'. The other implication is trivial. From m-dependence, if $j - i > m$, then

$$P(\|X_{ni}\| > \varepsilon,\ \|X_{nj}\| > \varepsilon) = P(\|X_{ni}\| > \varepsilon)\,P(\|X_{nj}\| > \varepsilon),$$

so due to (5.2) it is sufficient to show

$$\lim_{r\to\infty} \limsup_{n\to\infty} \inf_{\Delta(r)} \sum_{q=1}^{r} \sum_{i=m_{q-1}+1}^{m_q-m-1} P(\|X_{ni}\|>\varepsilon) \times\left(\sum_{j=i+m+1}^{m_q} P(\|X_{nj}\|>\varepsilon)\right)=0. \tag{5.3}$$

For every n and r, define a division of $\{1, ..., k_n\}$ as

$$m_0=0$$

$$m_q=\min\left\{k\geqslant m_{q-1};\ \sum_{i=1}^{k} P(\|X_{ni}\|>\varepsilon)>qc(\varepsilon)/r\right\},\quad q=1, ..., r.$$

We have for i such that $m_{q-1}<i\leqslant m_q$,

$$\sum_{j=i+m+1}^{m_q} P(\|X_{nj}\|>\varepsilon)\leqslant c(\varepsilon)/r+\max_{1\leqslant k\leqslant k_n} P(\|X_{nk}\|>\varepsilon)$$

and this together with A_0 and (5.1) gives (5.3). ∎

The following example shows that it is easy to find an array that is stationary and 1-dependent in rows, for which (5.2) is not true. For a stationary sequence $\{X_k;\,k\in Z\}$ we denote by $\{\hat{X}_k;\,k\in Z\}$ the "associated" sequence which consists of i.i.d. random vectors with $\mathscr{L}(\hat{X}_0)=\mathscr{L}(X_0)$.

EXAMPLE 5.2. Let $\{Y_k;\,k\in Z\}$ be an i.i.d. sequence of random variables such that $Y_0\geqslant 0$, $\mathscr{L}(Y_0)$, belong to the domain of attraction of the α-stable law $\rho(\alpha; 0, 1)$, $0<\alpha<2$. Define

$$X_k=\max(Y_k, Y_{k+1})$$

The sequence $\{X_k;\,k\in N\}$ is 1-dependent, $\mathscr{L}(X_0)$ belongs to the domain of attraction of an α-stable law μ_α by [18, Theorem 2], and hence the array

$$X_{nk}=X_k/a_n,\qquad 1\leqslant k\leqslant n,$$

where $\{a_n;\,n\in N\}$ is a sequence of norming constants such that $\sum_{k=1}^{n}\hat{X}_{nk}$ suitably centered tends in distribution to μ_α, satisfies A_0 and (5.1). However, D_0' is not fulfilled as

$$\sum_{q=1}^{r}\sum_{m_{q-1}<i<j\leqslant m_q} P(|X_{ni}|>\varepsilon, |X_{nj}|>\varepsilon)\geqslant (n-r)\,P(X_{n1}>\varepsilon, X_{n2}>\varepsilon)$$

$$\geqslant (n-r)\,P(Y_2>\varepsilon a_n)\xrightarrow[n\to\infty]{}\varepsilon^{-\alpha}.$$

Theorem 5.3 also concerns stationary and m-dependent sequences but this time it comprises the sequence from Example 5.2.

THEOREM 5.3. *Let $\{X_k;\ k\in Z\}$ be a strictly stationary, m-dependent sequence of d-dimensional random vectors, $d\in N$. Let $S_n=\sum_{k=1}^n X_k$, $n\in N$. Assume that the joint distribution of the random vector $(X_0, X_1, ..., X_m)=Y_0$ belongs to the domain of attraction of a nondegenerated $d(m+1)$-dimensional α-stable distribution $c_{\tau(\alpha)}-\mathrm{Pois}(\nu)$, $0<\alpha<2$; ($\tau(\alpha)$ is given in (2.6)). Let $\{a_n,\ n\in N\}$ be a sequence of reals such that*

> *the partial sums $(\hat{Y}_1+\cdots+\hat{Y}_n)/a_n$ suitably centered are convergent in distribution to μ (where $\hat{Y}_1, \hat{Y}_2, ...$ are the independent copies of Y_0).* (5.4)

Let

$$\nu_0 := \nu_{x_0+x_1+\cdots+x_m}-\nu_{x_1+\cdots+x_m} \tag{5.5}$$

(see definitions (2.8), (2.9)):

(i) *If $0<\alpha<1$ then $S_n/a_n \to_{\mathscr{D}} \mathrm{Pois}(\nu_0)$.*

(ii) *If $1<\alpha<2$ then $(S_n-ES_n)/a_n \to_{\mathscr{D}} c_\infty-\mathrm{Pois}(\nu_0)$.*

(iii) *If $\alpha=1$ then*

$$(S_n-nEX_0 1(\|X_0\|\leqslant a_n))/a_n \xrightarrow[\mathscr{D}]{} \delta_b * c_1-\mathrm{Pois}(\nu_0),$$

where

$$\begin{aligned} b=\int [&(x_0+\cdots+x_m)\,1(\|x_0+\cdots+x_m\|\leqslant 1)\\ &-(x_1+\cdots+x_m)\,1(\|x_1+\cdots+x_m\|\leqslant 1)\\ &-x_0 1(\|x_0\|\leqslant 1)]\,d\nu(x_0, ..., x_m). \end{aligned} \tag{5.6}$$

COROLLARY 5.4. *If we take in Theorem 5.1(iii) the centering vectors*

$$\begin{aligned} b_n &= nE(X_0+\cdots+X_m)\,1(\|X_0+\cdots+X_m\|\leqslant a_n)\\ &\quad -nE((X_1+\cdots+X_m)\,1(\|X_1+\cdots+X_m\|\leqslant a_n), \end{aligned} \tag{5.7}$$

then

$$(S_n-b_n)/a_n \xrightarrow[\mathscr{D}]{} c_1-\mathrm{Pois}(\nu_0).$$

The above result can be treated as the α-stable counterpart of the following central limit theorem [7].

THEOREM 5.5. *If $\{X_k;\ k \in N\}$ is a strictly stationary sequence of m-dependent random vectors with values in R^d such that $EX_0 = 0$, $E\|X_0\|^2 < +\infty$, then*

$$S_n/(n^{1/2}) \xrightarrow[\mathscr{D}]{} N(0, \Sigma),$$

where $S_n = \sum_{k=1}^{n} X_k$, $\sum = [\sigma_{ij}]$, $\sigma_{ij} = EX_0^i X_0^j + \sum_{k=1}^{m} E(X_0^i X_k^j + X_k^i X_0^j)$.

Notice that $\sum = \operatorname{Cov} S_{m+1} - \operatorname{Cov} S_m$. For a strictly stationary sequence of m-dependent random vectors $\{X_k; k \in Z\}$ denote by $\{\hat{X}_k^{(p)}; k \in Z\}$ the i.i.d. sequence of random vectors such that $\hat{X}_0^{(p)}$ has the same distribution as $\sum_{k=1}^{p} X_k$ and $\{\hat{X}_k^{(p)}; k \in Z\}$ is independent of $\{X_k; k \in Z\}$,

$$\hat{S}_n^{(p)} = \sum_{k=1}^{n} \hat{X}_k^{(p)}, \qquad S_n = \sum_{k=1}^{n} X_k.$$

Both results, Theorems 5.3 and 5.5 can be regarded as a partial answer to the following

Conjecture. Let $\{X_k;\ k \in Z\}$ be a strictly stationary and m-dependent sequence of random vectors. If the distributions $\mathscr{L}(S_{m+1})$ and $\mathscr{L}(S_m)$ belong to the domains of attraction of α-stable laws for some $\alpha \in]0, 2]$, then the sequences $\{\hat{S}_n^{(m)} + S_n;\ n \in N\}$ and $\{\hat{S}_n^{(m+1)}; n \in N\}$ identically normed and centered have the same limit in distribution.

We defer the technical proofs of Theorem 5.3 and Corollary 5.4 to Section 6. Here let us note only that the idea is based on Theorem 3.1 and is not so simple as in Section 4. Here the array

$$X_{nk} = X_k/a_n, \qquad k \in Z, n \in N$$

does not satisfy D′. Thus we make a special reduction and pass to a family of arrays $\{X_{nk}^{\eta**},\ k \in Z,\ n \in N\}_{\eta > 0}$ whose properties allow us to apply Theorem 3.1.

Below we consider two examples which give some information about the measure ν_0.

EXAMPLE 5.6. Let $\{Y_k; k \in Z\}$ be a two-sided sequence of i.i.d. random vectors and let $\{c_0, c_1, ..., c_m\}$ be a finite sequence of real numbers. We define a sequence of finite averages of $\{Y_k; k \in Z\}$

$$X_k = \sum_{j=0}^{m} c_j Y_{k-j}, \qquad k \in Z.$$

$\{X_k; k \in Z\}$ is a strictly stationary m-dependent sequence. We assume that the distribution $\mathscr{L}(Y_0)$ belongs to the domain of attraction of a stable distribution μ.

We find the measure ν_0: If U has the distribution μ, then $\mathscr{L}(X_0, X_1, ..., X_m)$ belongs to the domain of attraction of $\mathscr{L}(Z_0, Z_1, ..., Z_m)$, where $Z_i := \sum_{j=0}^{m} c_j U_{i-j}$, $i = 0, 1, ..., m$, and $\{U_k;\ k \in Z\}$ are independent copies of U. We have

$$\begin{aligned}
&\nu_{Z_0+Z_1+\cdots+Z_m}\\
&= \nu_{c_m U_{-m}+(c_m+c_{m+1})U_{-m+1}+\cdots+(c_m+c_{m-1}+\cdots+c_0)U_0+(c_{m-1}+c_{m-2}+\cdots+c_0)U_1+\cdots+c_0 U_m}\\
&= \nu_{c_m U_{-m}} + \nu_{(c_m+c_{m-1})U_{-m+1}} + \cdots + \nu_{(c_m+c_{m-1}+\cdots+c_0)U_0}\\
&\quad + \nu_{(c_{m-1}+c_{m-2}+\cdots+c_0)U_1} + \cdots + \nu_{c_0 U_1}\\
&= \nu_{c_m U} + \nu_{(c_m+c_{m-1})U} + \cdots + \nu_{(c_m+c_{m-1}+\cdots+c_0)U}\\
&\quad + \nu_{(c_{m-1}+c_{m-2}+\cdots+c_0)U} + \cdots + \nu_{c_0 U}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\nu_{Z_1+\cdots+Z_m} &= \nu_{c_m U} + \cdots + \nu_{(c_m+\cdots+c_1)U}\\
&\quad + \nu_{(c_{m-1}+\cdots+c_0)U} + \cdots + \nu_{c_0 U}.
\end{aligned}$$

Hence,

$$\nu_0 = \nu_{(c_m+c_{m+1}+\cdots+c_0)U}.$$

Here Theorem 5.4 (in the case of random variables) is a particular case of a general theorem about weak convergence of sums to stable limits for moving averages [6, Theorem 4.1].

EXAMPLE 5.7. Let $\{X_k;\ k \in N\}$ be a sequence of positive i.i.d. random variables such that $\mathscr{L}(X_1)$ belongs to the domain of attraction of the α-stable distribution $c_\tau - \text{Pois}(\nu)$, where $\nu = \nu(1; 0, 1)$ (i.e., $P(X_1 > x) = x^{-1}L(x)$, $x > 0$, and L is slowly varying in ∞). Let $\{a_k;\ k \in N\}$ be a sequence of constants such that

$$nP(X_1 > a_n x) \to x^{-1}, \qquad n \to \infty.$$

We define

$$Y_n = \max(X_n, 3X_{n+1}), \qquad n \in N.$$

The sequence has the properties:

(1) $\{Y_n;\ n \in N\}$ is stationary;

(2) $\{Y_n;\ n \in N\}$ is 1-dependent;

(3) the joint distribution $\mathscr{L}(Y_1, Y_2)$ belongs to the domain of attraction of a two-dimensional 1-stable distribution (by [18, Theory 2]);

(4) $nP(Y_1 > a_n, Y_2 > a_n) \to 1, n \to \infty$;

$$(5)\ nP(Y_1 > xa_n, Y_2 > ya_n) \to \begin{cases} ((x/3) \vee y)^{-1} & x>0, y>0, \\ x^{-1} + (x/3)^{-1} & x>0, y \leqslant 0, \\ y^{-1} + (y/3)^{-1} & x \leqslant 0, y>0, \end{cases}$$

(due to (3) and (4)).

The property (5) and limit theorem for the "associated" sequence $\{(\widehat{Y_k, Y_{k+1}}); k \in N\}$ imply that the measure ν is given by the formula:

$$\nu(]x, +\infty[\times]y, +\infty[) = \begin{cases} ((x/3) \vee y)^{-1} & x>0, y>0, \\ x^{-1} + (x/3)^{-1} & x>0, y \leqslant 0, \\ y^{-1} + (y/3)^{-1} & x \leqslant 0, y>0, \end{cases}$$

hence it is concentrated on the positive halflines and the positive part of the line $y = x/3$.

ν_η in Lemma 6.2 is given by the formula

$$\begin{aligned} \nu_\eta(A) = {} & \nu(x_0 \in A, |x_0| > \eta) + \nu(x_0 + x_1 \in A, |x_0| > \eta, |x_1| > \eta \\ & - \nu(x_0 \in A, |x_0| > \eta, |x_1| > \eta) \\ & - \nu(x_1 \in A, |x_1| > \eta, |x_0| > \eta). \end{aligned}$$

While $\eta \to 0$, then, due to Lemma 6.4,

$$\nu_\eta(A) \to \nu_0(A) = \nu(x_0 + x_1 \in A) - \nu(x_1 \in A)$$

for Borel sets A which are separated from 0 and

$$c_1 - \text{Pois } \nu_\eta \xrightarrow[w]{} c_1 - \text{Pois } \nu_0.$$

Moreover,

$$\begin{aligned} b &= \lim_{\eta \to 0} \left[\int x 1(|x| \leqslant 1)\, \nu_\eta(dx) - \int x 1(\eta < |x| \leqslant 1)\, \nu_{x_0}(dx) \right] \\ &= \lim_{\eta \to 0} \left\{ \int [(x_0 + x_1)\, 1(|x_0 + x_1| \leqslant 1, |x_0| > \eta, |x_1| > \eta) \right. \\ &\quad - x_0 1(|x_0| \leqslant 1, |x_0| > \eta, |x_1| > \eta) \\ &\quad \left. - x_1 1(|x_1| \leqslant 1, |x_0| > \eta, |x_1| > \eta)]\, \nu(dx) \right\} = 4 \ln 4 - 3 \ln 3. \end{aligned}$$

Notice that one cannot change the order of operations of taking the limit and integration.

6. Proofs

Lemma 6.1 gives an estimation which will be useful in the sequel.

Lemma 6.1. *If* $\{X_n=(X_n^1, ..., X_n^d);\ n\in N\}$ *is a sequence of random vectors, m-dependent, square integrable, and such that* $\mathscr{L}(X_n)=\mathscr{L}(X_1)$, *then there exists a constant* K *for which the inequality*

$$E\,\|S_n-ES_n\|^2\leqslant KnE\,\|X_1\|^2 \tag{6.1}$$

holds, where $S_n=\sum_{k=1}^n X_k$.

Lemma 6.2 is an essential step in the proof of Theorem 5.3 and is common for all the three cases.

Lemma 6.2. *Under the assumptions of Theorem* 5.3 *there holds the following convergence for every* $\eta>0$:

$$S_n(]\eta,+\infty[)=\sum_{k=1}^n X_k 1(\|X_k\|>\eta a_n)/a_n\to \mathrm{Pois}(\nu_n), \tag{6.2}$$

where

$$\nu_\eta(A)=\sum_{J\subset Z_m}\ \sum_{\phi\neq K\subset J\cup\{0\}}(-1)^{|J|+1-|K|}$$
$$\times\nu\left(\left\{(x_0,x_1,...,x_m);\ \sum_{k\in K}x_k\in A,\ \|x_j\|>\eta,\ j\in J\cup\{0\}\right\}\right) \tag{6.3}$$

with $Z_m=\{1,...,m\}$ *and Borel set* $A\subset R^d$ ($|U|$ *denotes power of the set* U).

Proof. The idea is based on Theorem 3.1; however, this result will be useful after a certain reduction.

Let us fix $\eta>0$ and define

$$X_{nk}=X_k/a_n,\quad X_{nk}^*=X_{nk}1_{B_{nk}},\quad S_n^*=\sum_{k=1}^n X_{nk}^*,$$

$$S_n]\alpha,\beta]=\sum_{k=1}^n X_{nk}1(\alpha<\|X_{nk}\|\leqslant\beta),$$

where

$$B_{nk}=\{\|X_{nk}\|>\eta\}\cap\left[\bigcup_{i=k}^{k+m}\left(\{\|X_{ni}\|>\eta\}\cap\bigcap_{j=1}^{m}\{\|X_{n,i+j}\|\leqslant\eta\}\right)\right].$$

By the definitions of X_{nk}^*, stationarity, and m-dependence of $\{X_k;\ k \in Z\}$, we have

$$\begin{aligned} P(S_n^* \neq S_n(]\eta, +\infty[)) &\leqslant nP(X_{n0}^* \neq X_{n0} 1(\|X_{n0}\| > \eta)) \\ &\leqslant nP\left(\{\|X_{n0}\| > \eta\} \cap \left[\bigcup_{j=1}^{m} \{\|X_{n,m+j}\| > \eta\}\right]\right) \\ &\leqslant nmP^2(\|X_{n0}\| > \eta) \to 0, \quad n \to \infty, \end{aligned} \tag{6.4}$$

since $nP(\|X_{n0}\| > \eta) \to \nu_{x_0}(\|x\| > \eta) < +\infty$ by (5.4). Let

$$\begin{aligned} X_{nk}^{**} &= \sum_{j=0}^{m} X_{n,k-j} 1(\|X_{n,k-j}\| > \eta) \\ &\quad \times 1(\|X_{nk}\| > \eta) \times 1(\|X_{n,k+i}\| \leqslant \eta, \quad i = 1, ..., m), \end{aligned} \tag{6.5}$$

$$S_n^{**} = \sum_{k=1}^{n} X_{nk}^{**}.$$

We have $S_n^{**} = S_n^*$ and by (6.4) it is sufficient to show

$$S_n^{**} \xrightarrow[\mathscr{D}]{} \text{Pois}(\nu_n), \qquad n \to \infty.$$

In the lemma below we recapitulate the properties of $\{X_{nk}^{**}\}$.

LEMMA 6.3. *The array $\{X_{nk}^{**};\ k \in Z,\ n \in N\}$ satisfies the conditions*:

(a) *$\{X_{nk}^{**}\}$ is $(3m+1)$-dependent in rows;*

(b) *for every Borel set A in E^d such that $0 \notin \bar{A}$ and for every nonempty subset K of $Z_m \cup \{0\}$*

$$\nu_{\sum_{k \in K} x_k}(\partial A) = 0$$

there holds the convergence

$$nP(X_{n0}^{**} \in A) \to \nu_\eta(A), \quad n \to \infty; \tag{6.6}$$

(c) *for every $\varepsilon > 0$,*

$$\lim_{k \to \infty} \limsup_{n \to \infty} k \sum_{0 < i < j \leqslant [n/k]} P(\|X_{ni}^{**}\| > \varepsilon,\ \|X_{nj}^{**}\| > \varepsilon) = 0 \tag{6.7}$$

(d) $$\lim_{\delta \to 0} \limsup_{n \to \infty} nE\, \|X_{n0}^{**}\|\, 1(\|X_{n0}^{**}\| \leqslant \delta) = 0. \tag{6.8}$$

Proof of Lemma 6.3. (a) is trivial.

(c) Notice that

$$\{\|X_{ni}^{**}\|>\varepsilon\}\cap\{\|X_{nj}^{**}\|>\varepsilon\}=\phi \qquad \text{if} \quad j-i\leqslant m$$

and

$$P(\|X_{ni}^{**}\|>\varepsilon,\ \|X_{nj}^{**}\|>\varepsilon)\leqslant P^2(\|X_{n0}\|>\varepsilon) \qquad \text{if} \quad j-i>m.$$

Hence

$$\lim_{k\to\infty}\limsup_{n\to\infty} k \sum_{0<i<j\leqslant[n/k]} P(\|X_{ni}^{**}\|>\varepsilon,\ \|X_{nj}^{**}\|>\varepsilon)=0.$$

(d) is true because of (5.4) and the inequality

$$\|X_{n0}^{**}\|\ 1(X_{n0}^{**}\|\leqslant\delta)\leqslant\delta 1(\|X_{n0}\|>\eta).$$

The proof of (b) is more complicated. Notice that if $0\notin A$, then

$$\begin{aligned}1(X_{nm}^{**}\in A)&=\sum_{m\in J\subset Z_m\cup\{0\}} 1\left(\sum_{j\in J}X_{nj}\in A\right)\\&\quad\times 1(\|X_{nj}\|>\eta,\ j\in J)\times 1(\|X_{nk}\|\leqslant\eta,\ k\notin J,\ 0\leqslant k\leqslant 2m)\\&=\sum_{m\in J\subset Z_m\cup\{0\}}\ \sum_{K\subset(Z_{2m}\cup\{0\})\setminus J}(-1)^{|K|}\,1\left(\sum_{j\in J}X_{nj}\in A\right)\\&\quad\times 1(\|X_{nj}\|>\eta,\ j\in J\cup K).\end{aligned}$$

If $\max(J\cup K)-\min(J\cup K)>m$ then by m-dependence of $\{X_k, k\in Z\}$ and (5.4),

$$\begin{aligned}&nP\left(\sum_{j\in J}X_{nj}\in A,\ \|X_{nj}\|>\eta,\ j\in J\cup K\right)\\&\qquad\leqslant nP^2(\|X_{n0}\|>\eta)\to 0, \quad n\to\infty.\end{aligned}$$

Hence by stationarity it is sufficient to find the limit of the expression

$$\begin{aligned}&\sum_{J\subset Z_m}\ \sum_{\varnothing\neq K\subset J\cup\{0\}}(-1)^{|J|+1-|K|}\\&\qquad\times nP\left(\sum_{k\in K}X_{nk}\in A,\ \|X_{nj}\|>\eta,\ j\in J\cup\{0\}\right).\end{aligned}\tag{6.9}$$

Let $\{Y(s)=(x_0(s),\ x_1(s),\ldots,x_m(s)),\ s\in[0,1]\}$ be a $d(m+1)$-dimensional homogeneous process with independent increments such that $Y(0)=0$,

$\mathscr{L}(Y(1)) = c_{\tau(\alpha)} - \mathrm{Pois}(\nu)$. Then ν is a jump measure of Y, i.e., for every Borel set $A \subset R^{d(m+1)} \setminus \{0\}$,

$$\nu(A) = E\left(\sum_{s \leqslant 1} 1(\Delta Y(s) \in A)\right), \tag{6.10}$$

where $\Delta Y_s = Y_s - Y_{s-}$.

Due to (5.4) for every $J \subset Z_m$, $K \subset J \cup \{0\}$, $K \neq \emptyset$,

$$\begin{aligned} nP&\left(\sum_{k \in K} X_{nk} \in A,\ \|X_{nj}\| > \eta,\ j \in J \cup \{0\}\right) \\ &\to \nu\left(\sum_{k \in K} x_k \in A,\ \|x_j\| > \eta,\ j \in J \cup \{0\}\right) \\ &= E\left(\sum_{s \leqslant 1} 1\left(\sum_{k \in K} \Delta x_k(s) \in A,\ \|\Delta x_j(s)\| > \eta,\ j \in J \cup \{0\}\right)\right) \end{aligned} \tag{6.11}$$

if only $0 \notin \bar{A}$ and

$$\nu_{\sum_{k \in K} x_k}(\partial A) = 0. \tag{6.12}$$

(6.9) and (6.11) give (6.6). ▮

Now the thesis of Lemma 6.2 follows from Theorem 3.1 applied to the array $\{X^{**}_{nk}, k \in Z, n \in N\}$. ▮

Lemma 6.4 shows the asymptotic properties of the measure ν_η when $\eta \to 0$.

LEMMA 6.4. *For $\eta \to 0$ we have*

(i) $\nu_\eta(A) \to \nu_0(A) = \nu_{x_0 + x_1 + \cdots + x_m}(A) - \nu_{x_1 + x_2 + \cdots + x_m}(A)$ *for every Borel set A such that $0 \notin \bar{A}$,*

$$\text{(ii)} \qquad c_{\tau(\alpha)} - \mathrm{Pois}(\nu_\eta) \xrightarrow[w]{} c_{\tau(\alpha)} - \mathrm{Pois}(\nu_0), \quad 0 < \alpha < 2. \tag{6.13}$$

Proof. Let $J \subset \{1, ..., m\}$, $K \subset J \subset \{0\}$, $K \neq \emptyset$. Since A is separated from 0, hence there exists a finite limit

$$\begin{aligned} \lim_{\eta \to 0} \nu&\left(\sum_{k \in K} x_k \in A,\ \|x_j\| > \eta,\ j \in J \cup \{0\}\right) \\ &= \nu\left(\sum_{k \in K} x_k \in A,\ \|x_j\| > 0,\ j \in J \cup \{0\}\right). \end{aligned}$$

Thus also $\nu_\eta(A)$ is convergent when $\eta \to 0$.

In order to obtain the simple form of the limit we use the representation of the measure ν as a jump measure for the process $\{Y(s), s \in [0, 1]\}$. Since there exists a version of Y which has right-continuous trajectories and admits left-hand limits, so by the Lebesgue dominated theorem,

$$\lim_{\eta \to 0} E\left(\sum_{s \leq 1} 1\left(\sum_{k \in K} \Delta x_k(s) \in A,\ \|\Delta x_j(s)\| > \eta,\ j \in J \cup \{0\} \right)\right)$$

$$= E\left(\sum_{s \leq 1} 1\left(\sum_{k \in K} \Delta x_k(s) \in A,\ \|\Delta x_j(s)\| > 0,\ j \in J \cup \{0\} \right)\right) \quad (6.14)$$

(as a dominating function we can take

$$\sum_{s \leq 1} 1(\Delta(C_K \circ Y(s)) \in A),$$

where $C_K\colon (R^d)^{m+1} \to R^d$,

$$C_K(x_0, ..., x_m) = \sum_{k \in K} x_k.$$

Hence to prove (i) it remains to be shown that the measure

$$\nu_{x_0 + x_1 + \cdots + x_m}(\cdot) - \nu_{x_1 + x_2 + \cdots + x_m}(\cdot)$$

$$= E\left[\sum_{s \leq 1} (1(\Delta(C_{Z_m \cup \{0\}} \circ Y(s)) \in \cdot) - i(\Delta(C_{Z_m} \circ Y(s)) \in \cdot)) \right] \quad (6.15)$$

$$E\left[\sum_{J \subset Z_m} \sum_{\varnothing \neq K \subset J \cup \{0\}} (-1)^{|J| + 1 - |K|} \left(\sum_{s \leq 1} 1\left(\sum_{k \in K} \Delta x_k(s) \in \cdot,\ \|\Delta x_j(s)\| > 0,\ j \in J \cup \{0\} \right)\right)\right]. \quad (6.16)$$

In order to obtain this we now prove the following technical lemma.

LEMMA 6.5. *Let $W = Z_m \cup \{0\}$. If the functions $f\colon 2^W \to R^1$, $g\colon 2^W \to R^1$ are such that*

1. $f(\varnothing) = 0$.
2. $f(K \cup J) \prod_{j \in J} g(j) = f(K) \prod_{j \in J} g(j)$ *if* $K, J \subset W$, $K \cap J = \varnothing$,

where, by convention, $\prod_{j \in J} g(j) = 1$ *if* $J = \varnothing$, $g(j) := g(\{j\})$, *then*

$$\sum_{J \subset W \setminus \{0\}} \sum_{\varnothing \neq K \subset J \cup \{0\}} (-1)^{|J| + 1 - |K|} f(K) \prod_{j \in J \cup \{0\}} (1 - g(j))$$

$$= f(W) - f(W \setminus \{0\}). \quad (6.17)$$

Proof. Since $f(\varnothing) = 0$, so the left-hand side expression (L) in (6.17) can be completed by the summands depending on $K = \varnothing$. Thus we have

$$
\begin{aligned}
\mathrm{L} &= \sum_{J \subset W\setminus\{0\}} \sum_{K \subset J \cup \{0\}} (-1)^{|J|+1-|K|} f(K) \prod_{j \in J \cup \{0\}} (1-g(j)) \\
&= \sum_{K \subset W} \sum_{K\setminus\{0\} \subset J \subset W\setminus\{0\}} (-1)^{|J|+1-|K|} f(K) \prod_{j \in J \cup \{0\}} (1-g(j)) \\
&= \sum_{0 \in K \subset W} \sum_{J' \subset W\setminus K} (-1)^{|J'|} f(K) \prod_{j \in J' \cup K} (1-g(j)) \\
&\quad + \sum_{K \subset W\setminus\{0\}} \sum_{J' \in (W\setminus\{0\})\setminus K} (-1)^{|J'|+1} f(K) \prod_{j \in J' \cup K \cup \{0\}} (1-g(j)) \\
&= \sum_{0 \in K \subset W} f(K) \prod_{j \in K} (1-g(j)) \sum_{J' \subset W\setminus K} (-1)^{|J'|} \prod_{j \in J'} (1-g(j)) \\
&\quad - (1-g(0)) \sum_{K \subset W\setminus\{0\}} f(K) \prod_{j \in K} (1-g(j)) \\
&\quad \times \sum_{J' \in (W\setminus\{0\})\setminus K} (-1)^{|J'|} \prod_{j \in J'} (1-g(j)) \\
&= \sum_{0 \in K \subset W} f(K) \prod_{j \in K} (1-g(j)) \prod_{j \in W\setminus K} g(j) \\
&\quad - (1-g(0)) \sum_{K \subset W\setminus\{0\}} f(K) \prod_{j \in K} (1-g(j)) \prod_{j \in (W\setminus\{0\})\setminus K} g(j) \\
&= \sum_{0 \in K \subset W} f(W) \prod_{j \in K} (1-g(j)) \prod_{j \in W\setminus K} g(j) \\
&\quad - (1-g(0)) \sum_{K \subset W\setminus\{0\}} f(W\setminus\{0\}) \\
&\quad \times \prod_{j \in K} (1-g(j)) \prod_{j \in (W\setminus\{0\})\setminus K} g(j) \\
&= f(W)(1-g(0)) \\
&\quad \times \sum_{K \subset W\setminus\{0\}} \prod_{j \in K} (1-g(j)) \prod_{j \in (W\setminus\{0\})\setminus K} g(j) \\
&\quad - f(W\setminus\{0\})(1-g(0)) \\
&\quad \times \sum_{K \subset W\setminus\{0\}} \prod_{j \in K} (1-g(j)) \prod_{j \in (W\setminus\{0\})\setminus K} g(j) \\
&= f(W)(1-g(0)) - f(W\setminus\{0\})(1-g(0)) \\
&= f(W) - f(W\setminus\{0\}). \quad \blacksquare
\end{aligned}
$$

Now in Lemma 6.5 taking

$$f(K)=1\left(\sum_{k\in K} x_k \in A\right) \qquad f\colon 2^{Z_m\cup\{0\}}\to R$$

$$g(J)=1(\|x_j\|=0,\ j\in J) \qquad g\colon 2^{Z_m\cup\{0\}}\to R,$$

we have

$$\sum_{J\subset Z_m}\ \sum_{\varnothing\neq K\subset J\cup\{0\}} (-1)^{|J|+1-|K|}\,1\left(\sum_{k\in K} x_k\in A\right)$$
$$\times 1(\|x_j\|>0,\ j\in J\cup\{0\})$$
$$=1\left(\sum_{k=0}^{m} x_k\in A\right)-1\left(\sum_{k=1}^{m} x_k\in A\right).$$

Hence, really, the expressions in (6.15) and (6.16) are equal and this ends the proof of (i).

By the criteria for infinitely divisible distributions (see [2]) for proving (ii), it is sufficient to show that

$$\lim_{\delta\to 0}\ \limsup_{\eta\to 0}\int \|x\|^2\,1(\|x\|\leqslant\delta)\,\nu_\eta(dx)=0.$$

Due to (6.3), the above integral is dominated by a finite number of integrals of the form

$$\int \|x\|^2\,1(\|x\|\leqslant\delta)\,\nu_{\sum_{k\in K}x_k}(dx), \qquad K\subset Z_m\cup\{0\}$$

which tend to 0 while $\delta\to 0$, since $\nu_{\sum_{k\in K}x_k}$ is a Lévy measure. The proof of Lemma 6.4 is finished. ∎

Now we are ready to continue the

Proof of Theorem 5.4. The idea is based on [3, Theorem 4.2]. Let

$$X_{nk}=X_k/a_n.$$

For every $\eta>0$ we break the expression

$$S_n/a_n - nEX_{n0}1(\|X_{n0}\|\leqslant\tau(\alpha))$$

into the sum of three summands:

$$\begin{aligned} S_n/a_n - nEX_{n0}1(\|X_{n0}\|\leqslant\tau(\alpha)) &= [S_n([0,\eta]) - ES_n([0,\eta\wedge\tau(\alpha)])] \\ &\quad + S_n(]\eta,+\infty[) - ES_n(]\eta\wedge\tau(\alpha),\tau(\alpha)]) \end{aligned} \tag{6.18}$$

where $S_n(]a,b])=\sum_{k=1}^{n} X_{nk}1(a<\|X_{nk}\|\leqslant b)$.

By Lemma 6.2 for fixed $\eta > 0$,

$$S_n(]\eta, +\infty[) \to \operatorname{Pois}(\nu_\eta), \qquad n \to +\infty. \tag{6.19}$$

Since the law of X_0 belongs to the domain of attraction of the distribution $c_{\tau(\alpha)} - \operatorname{Pois}(\nu_{x_0})$, so

$$ES_n(]\eta \wedge \tau(\alpha), \tau(\alpha)]) \to \int x1(\eta \wedge \tau(\alpha) < \|x\| \leqslant \tau(\alpha))\, \nu_{x_0}(dx) = b_\eta \tag{6.20}$$

as $n \to \infty$.

It remains to show that that for every $\varepsilon > 0$,

$$\lim_{\eta \to 0} \limsup_{n \to \infty} P(\|S_n(]0, \eta]) - ES_n([0, \eta \wedge \tau(\alpha)])\| > \varepsilon) = 0 \tag{6.21}$$

and that if $\eta \to 0$, then

$$\left\{\int x1(\|x\| \leqslant \tau(\alpha))\, \nu_\eta(dx) - b_\eta ; \eta > 0\right\} \tag{6.22}$$

is convergent to some $b(\alpha) \in R^d$, since then by (6.13),

$$\operatorname{Pois}(\nu_\eta) * \delta_{-b_\eta} \xrightarrow[w]{} c_{\tau(\alpha)} - \operatorname{Pois}(\nu_0) * \delta_{b(\alpha)}. \tag{6.23}$$

We consider each of the three cases separately.

I. $0 < \alpha < 1$. Since the law of X_0 belongs to the domain of attraction of $\operatorname{Pois}(\nu_{x_0})$, hence

$$\lim_{\eta \to 0} \limsup_{n \to \infty} nE \|X_{n0}\| \, 1(\|X_{n0}\| \leqslant \eta) = 0.$$

This implies (6.21). As $\tau(\alpha) = 0$, (6.22) is obvious: $b_\eta = b(\alpha) = 0$ and

$$\operatorname{Pois}(\nu_\eta) \xrightarrow[w]{} \operatorname{Pois}(\nu_0).$$

II. $1 < \alpha < 2$. Applying Lemma 6.1 to the random vectors $X_{nk} 1(\|X_{nk}\| \leqslant \eta)$, $1 \leqslant k \leqslant n$, we have

$$E \|S_n([0, \eta]) - ES_n([0, \eta]))\|^2 \leqslant KnE \|X_{n0}\|^2 \, 1(\|X_{n0}\| \leqslant \eta).$$

By (5.4),

$$\lim_{\eta \to 0} \limsup_{n \to \infty} nE \|X_{n0}\|^2 \, 1(\|X_{n0}\| \leqslant \eta) = 0,$$

so (6.21) is satisfied.

Now we show that

$$\int x\nu_\eta(dx)=\int x1(\|x\|\geqslant\eta)\,\nu_{x_0}(dx)\qquad(=b_\eta). \tag{6.24}$$

Using (6.3) and the representation (6.10), we obtain

$$\begin{aligned}\nu_\eta(A)=\sum_{J\subset Z_m}\ \sum_{\varnothing\neq K\subset J\cup\{0\}}&(-1)^{|J|+1-|K|}\\ &\times\nu\left(\sum_{k\in K}x_k\in A,\ \|x_j\|>\eta,\ j\in J\cup\{0\}\right)\end{aligned} \tag{6.25}$$

and thus

$$\begin{aligned}\int x\nu_\eta(dx)=E\Bigg[\sum_{J\subset Z_m}(-1)^{|J|+1}&\sum_{\varnothing\neq K\subset J\cup\{0\}}(-1)^{-|K|}\\ &\times\sum_{s\leqslant 1}\left(\sum_{k\in K}\Delta x_k(s)\right)1(\|\Delta x_j(s)\|>\eta,\ j\in J\cup\{0\})\Bigg].\end{aligned} \tag{6.26}$$

If $J\neq\varnothing$, then for every $s\in[0,1]$,

$$\begin{aligned}&\sum_{\varnothing\neq K\subset J\cup\{0\}}(-1)^{-|K|}\left(\sum_{k\in K}\Delta x_k(s)\right)\\ &\quad=\sum_{k\in J\cup\{0\}}\left(\sum_{\substack{K\subset J\cup\{0\}\\ k\in K}}(-1)^{|K|}\right)(\Delta x_k(s))=0.\end{aligned}$$

If $J=\varnothing$ then the respective expression in (6.26) is equal to

$$E\left[\sum_{s\leqslant 1}\Delta x_0(s)\,1(\|\Delta x_0(s)\|>\eta)\right]=\int x1(\|x\|\geqslant\eta)\,\nu_{x_0}(dx).$$

Hence (6.24) is true. Thus we have $b(\alpha)=0$ and hence (ii) is proved.

III. $\alpha=1$. The condition (6.21) can be proved as for $1<\alpha<2$. We have to find the limit

$$b=\lim_{\eta\to 0}\left[\int x1(\|x\|\leqslant 1)\,\nu_\eta(dx)-\int x1(\eta<\|x\|\leqslant 1)\,\nu_{x_0}(dx)\right] \tag{6.27}$$

or, taking an obvious modification of (6.26), we should prove that for every $J\neq\varnothing$, $J\subset\{1,2,\ldots,m\}$, there exists

$$\lim_{\eta \to 0} E\left\{\sum_{s \leqslant 1}\left[\sum_{\varnothing \neq K \subset J \cup \{0\}} (-1)^{-|K|}\left(\sum_{k \in K} \Delta x_k(s)\right) 1\left(\left\|\sum_{k \in K} \Delta x_k(s)\right\| \leqslant 1\right)\right]\right.$$
$$\left. \times 1(\|\Delta x_j(s)\| > \eta,\ j \in J \cup \{0\})\right\} \tag{6.28}$$

and find it.

But the expression in the square brackets in (6.28) is equal to 0, if for all K such that $\varnothing \neq K \subset J \cup \{0\}$, we have $\sum_{k \in K} \|\Delta x_k(s)\| \leqslant 1$. Hence it can be multiplied by

$$1_{B(J)} = 1\left(\left\|\sum_{k \in K} \Delta x_k(s)\right\| > 1 \text{ for some } K \subset J \cup \{0\},\ K \neq \varnothing\right).$$

This allows us to apply the Lebesgue dominated theorem and find limits in (6.28) and (6.27).

We have

$$b + \int x 1(0 < \|x\| \leqslant 1)\, \nu_{x_0}(dx)$$
$$= E\left\{\sum_{s \leqslant 1}\left[\sum_{\varnothing \neq J \subset Z_m} (-1)^{|J|+1} \sum_{\varnothing \neq K \subset J \cup \{0\}} (-1)^{-|K|}\right.\right.$$
$$\times \left(\sum_{k \in K} \Delta x_k(s)\right) 1\left(\left\|\sum_{k \in K} \Delta x_k(s)\right\| \leqslant 1\right)$$
$$\left.\left. \times 1(\Delta x_j(s) \neq 0,\ j \in J \cup \{0\})\right]\right\}$$
$$= \int [(x_0 + x_1 + \cdots + x_m) 1(\|x_0 + x_1 + \cdots + x_m\| \leqslant 1)$$
$$- (x_1 + x_2 + \cdots + x_m) 1(\|x_1 + x_2 + \cdots + x_m\| \leqslant 1)]$$
$$\times d\nu(x_0, \ldots, x_m). \quad \blacksquare$$

Proof of Corollary 5.4. It is sufficient to show that

$$nE\left[\left(\sum_{k=0}^{m} X_{nk}\right) 1\left(\left\|\sum_{k=0}^{m} X_{nk}\right\| \leqslant 1\right) - \left(\sum_{k=1}^{m} X_{nk}\right) 1\left(\left\|\sum_{k=1}^{m} X_{nk}\right\| \leqslant 1\right)\right.$$
$$\left. - X_{n0} 1(\|X_{n0}\| \leqslant 1)\right] \to b, \quad n \to \infty \text{ (see 6.27).}$$

But the function under the integral is equal to 0 on the intersection

$$\left\{\left\|\sum_{k=0}^{m} X_{nk}\right\| \leqslant 1\right\} \cap \left\{\left\|\sum_{k=1}^{m} X_{nk}\right\| \leqslant 1\right\} \cap \{\|X_{n0}\| \leqslant 1\}$$

and hence the support of the function

$$f(x_0, ..., x_m) = \sum_{k=0}^{m} x_k 1\left(\left\|\sum_{k=0}^{m} x_k\right\| \leqslant 1\right)$$

$$- \sum_{k=1}^{m} x_k 1\left(\left\|\sum_{k=1}^{m} X_k\right\| \leqslant 1\right) - x_0 1(\|x_0\| \leqslant 1)$$

$$f\colon (R^d)^{m+1} \to R^d$$

is separated from 0. If we denote by $P_{(X_{n0}, ..., X_{nm})}$ the distribution of $(X_{n0}, ..., X_{nm})$ then due to (5.4),

$$\int f(x_0, ..., x_m)\, d(nP_{(X_{n0}, ..., X_{nm})}) \to \int f(x_0, ..., x_m)\, dv. \quad \blacksquare$$

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